Geometric properties of passive random advection

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Geometric properties of a random Gaussian short-time correlated velocity field are studied by considering the statistics of a passively advected metric tensor. That describes the universal properties of the fluctuations of tensor objects frozen into the fluid and passively advected by it. The problem of the one-point statistics of covariant and contravariant tensors is solved exactly, provided that the advected fields do not reach diffusive scales, which would break the symmetry of the problem.

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I. INTRODUCTION

The problem of passive advection in a turbulent medium has attracted considerable attention as a solvable model of turbulence. Exact solutions can be found in the simplified case in which the velocity field is chosen to be a random, short-time correlated Gaussian process. The statistics of density, concentration, and passive vectors advected by such a field were investigated by many authors (see, e.g., [1–9]), and the intermittent nature of the fluctuations, nontrivial scalings of structure functions, and the anomalous role of the dissipation were discovered. All these features are very common in the general picture of turbulence, so the problem of passive advection can serve as a model for developing appropriate analytical tools.

In the present paper we consider passive advection (in the Lie sense) of a second-rank covariant tensor in d-dimensional space. Though the master equation for the probability distribution function (PDF) of the tensor [Eq. (5) below] is very general, we concentrate mainly on the statistics of a symmetric (metric) tensor g_{ij}. The one-point statistics of any tensor object frozen into the fluid can be related to the statistics of such a tensor. We do not impose any restrictions (such as incompressibility) on the velocity field; therefore, the statistics in both Eulerian and Lagrangian frames can be studied. Also, we are only interested in the ''initial stage'' of the advection, when the advected field does not reach diffusion scales. This allows one to explore the symmetries of the problem; those are broken when diffusion is included. In this context, two physical remarks are in order, before we proceed with formal mathematical consideration.

First, the problem that is considered in this paper is generally referred to as ''*fast*,'' or ''*kinematic*,'' *dynamo* in its application to the randomly advected magnetic fields. It is relevant for physical settings where typical scales of the advecting velocity field are much larger than typical diffusion scales of the advected passive fields. The astrophysics of interstellar medium and protogalactic plasmas are good examples of such applications. In particular, for magnetic fields generated in the galaxy or protogalaxy, the ratio R of the velocity-field scale to the resistive (diffusive) scale, ranges from $R\sim 10^8$ to $R\sim 10^{12}$. The magnetic fluctuations are generated at the scale of the velocity field and propagate in k space towards the (small) diffusive scales. It is important that the time the fluctuations of the passive field take to reach the diffusive scales is large when R is large. To illustrate this point with an example, consider the spectral density $M(t,k)=\langle|\mathbf{B}(t,\mathbf{k})|^2\rangle$ of some advected vector field $\mathbf{B}$. Straightforward calculation shows that it evolves in k space according to the following diffusion-type equation [2,10,11]:

$$\partial_t M = A_d(a)k^2\frac{\partial^2}{\partial k^2}M + B_d(a)k\frac{\partial}{\partial k}M + C_d(a)M. \quad (1)$$

One can easily show that this evolution equation is universal: i.e., an equation of this form holds for the spectral density of *any* advected tensor [11]. The coefficients A, B, and C depend on the space dimension d, the compressibility parameter a of the velocity field [Eq. (3) below], and the tensor structure of the advected field. The diffusive scales are reached at times $t\sim\log(R)$.

Second, we consider the limit of short-time correlated velocity field: i.e., the correlation time is assumed to be small compared to the inverse growth rates of the advected fields. In this limit, the growth rates that we obtain for one-point objects are independent of the particular spacial structure of the velocity correlation function. For arbitrary correlation times, such universality is lost. The latter case falls beyond the scope of the present paper. We refer the reader to Refs. [8], [9], [12] and references therein for discussions of such a regime.

We show that the PDF of the eigenvalues of the metric $\hat{g}$ is governed by a d-particle Hamiltonian that can be split into two noninteracting parts. Its *nonuniversal* part describes the motion of the center of mass (the determinant g of the metric) and can be separated from the motion relative to the center of mass, i.e., the dynamics of the metric's eigenvalues normalized to their geometrical mean, $\lambda_i/g^{1/d}$. The Hamiltonian of the latter motion is of the Calogero-Sutherland type, remains the same in both Lagrangian and Eulerian frames of reference, and therefore describes the *universal* properties of the advection. These properties are dictated by the symmetries of the problem. The exact integrability of the Calogero-Sutherland Hamiltonian is known to be related to SL(d) symmetry: the Hamiltonian can be represented as a

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quadratic polynomial in terms of the generators of the corresponding algebra [7,13,14]. The eigenfunctions of this Hamiltonian are the so-called Jack polynomials, which are symmetric homogeneous functions of the eigenvalues. This allows one to find exactly all moments $\langle \hat{T}^m \rangle$ of any tensor $\hat{T}$ advected by the fluid. Indeed, calculating any such moment reduces to averaging expressions of the type $\mathrm{Tr}^k(\hat{g}^n)$ (where ''Tr'' denotes trace), which are symmetric polynomials in terms of the metric's eigenvalues and can therefore be expanded in Jack polynomials of degree *nk*. We illustrate this method by calculating exactly all moments of passively advected vectors and covectors—in particular, of the magnetic field in the kinematic regime and of the passive-scalar gradient. We also demonstrate how this approach works in the general case of a passively advected tensor of any rank.

Calculating the moments requires knowing the statistics of the metric $\hat{g}$ with special initial conditions $g_{ij}(t=0) = \delta_{ij}$. However, it is also interesting to consider the evolution of the PDF of the symmetric tensor g_{ij} subject to arbitrary initial conditions. In this context, we show that a beautiful dual picture exists: the time-dependent PDF of the tensor becomes asymptotically ($t\to\infty$) invariant under the inversion of the eigenvalues with respect to their geometrical mean. For example, in three dimensions, that means that if a magnetic field advected by ideally conducting fluid develops flux tubes, it must develop magnetic sheets with the same probability.

The paper is organized as follows. In Sec. II, we derive the master equation for the PDF of the metric's eigenvalues and analyze the symmetry properties of that PDF. In Sec. III, we present a simple method of transforming the PDF between Eulerian and Lagrangian frames, which is important in the case of a compressible velocity field. In Sec. IV, we discuss general properties of solutions for the PDF in the two- and three-dimensional cases. In Secs. V and VI, we show how the symmetry of the problem allows one to calculate all the moments of passively advected tensors. The article is written in a self-contained manner; all the necessary definitions and derivations are summarized in the Appendixes.

II. MASTER EQUATION FOR THE PDF OF METRIC TENSOR

A covariant second-rank tensor $\varphi_{ij}(t,\mathbf{x})$ passively advected by the velocity field $\xi^k(t,\mathbf{x})$ evolves according to the equation

$$\partial_t \varphi_{ij} + \xi^k \varphi_{ij,k} + \xi^k_{,i}\varphi_{kj} + \xi^k_{,j}\varphi_{ik} = 0, \tag{2}$$

where $\xi^k_{,i} = \partial\xi^k/\partial x^i$ and $\varphi_{ij,k} = \partial\varphi_{ij}/\partial x^k$. Let $\xi^i(t,\mathbf{x})$ be a Gaussian-Markov field:

$$\langle \xi^i(t,\mathbf{x})\xi^j(t',\mathbf{x}')\rangle = \kappa^{ij}(\mathbf{x}-\mathbf{x}')\,\delta(t-t'),$$

$$\kappa^{ij}(\mathbf{y}) \simeq \kappa_0\delta^{ij} - \kappa_2(y^2\delta^{ij} + 2a y^i y^j), \quad y\to 0, \tag{3}$$

where a is the compressibility parameter and $\kappa_2 = 1$ for simplicity. Here a can vary between $-1/(d+1)$ for incompressible flow and 1 for irrotational flow.

In order to determine the statistics of the tensor, we follow a standard procedure [15,16] and introduce the characteristic function of $\varphi_{ij}(t,\mathbf{x})$:

$$Z(t,\hat{\sigma}) = \langle \exp[i\sigma^{ij}\varphi_{ij}(t,\mathbf{x})]\rangle. \tag{4}$$

This function is the Fourier transform of the PDF of the tensor elements φ_{ij}. Clearly, Z is independent of $\mathbf{x}$ due to spacial homogeneity. One finds that Z satisfies

$$\partial_t Z = -[1+a(d+1)]\sigma^{ij}\frac{\partial Z}{\partial\sigma^{ij}} + 2a\sigma^{ij}\frac{\partial}{\partial\sigma^{ij}}\sigma^{mn}\frac{\partial Z}{\partial\sigma^{mn}} + \frac{1}{2}\left(\sigma^{ij}\frac{\partial}{\partial\sigma^{kj}} + \sigma^{ji}\frac{\partial}{\partial\sigma^{jk}}\right)\left(\sigma^{il}\frac{\partial}{\partial\sigma^{kl}} + \sigma^{li}\frac{\partial}{\partial\sigma^{lk}} + a\sigma^{kl}\frac{\partial}{\partial\sigma^{il}} + a\sigma^{lk}\frac{\partial}{\partial\sigma^{li}}\right)Z, \tag{5}$$

where d is the dimensionality of space. This equation was derived by taking the time derivative of Z, using Eq. (2), and splitting Gaussian averages. One obtains the equation for the PDF of φ_{ij} by Fourier transforming Eq. (5):

$$P(\hat{\varphi}) = \int \exp(-i\sigma^{ij}\varphi_{ij})Z(\hat{\sigma})\prod_{m,n} d\sigma^{mn}. \tag{6}$$

The original equation (2) preserves the symmetry properties of the tensor φ_{ij}, which, means that one may restrict consideration to advection of either symmetric or antisymmetric tensors. Both reductions can be done in a similar fashion. For present purposes we only consider fluctuations of a symmetric covariant tensor. The corresponding results for a contravariant tensor are summarized in Appendix A. We will use both (covariant and contravariant) pictures when discussing statistics of passive vectors in Sec. V. The original version of this work appeared in Ref. [24].

In the symmetric case, the PDF (6) can be factorized as follows:

$$P(\hat{\varphi}) = \tilde{P}(\hat{g})\prod_{m<n}\delta(\varphi_{mn}-\varphi_{nm}), \tag{7}$$

where $\hat{g}$ is the symmetric part of $\hat{\varphi}$. One may think of the tensor g_{ij} as a metric associated with the medium. Due to spatial isotropy, $\tilde{P}$ depends only on the eigenvalues $\lambda_1,\ldots,\lambda_d$ of $\hat{g}$. After rather cumbersome but essentially simple calculations, one establishes the following master equation for the PDF of the eigenvalues of the metric:

$$\partial_t P = 2(2a+1)\sum_i \lambda_i^2\frac{\partial^2 P}{\partial\lambda_i^2} + 2a\sum_{i\neq j}\lambda_i\lambda_j\frac{\partial^2 P}{\partial\lambda_i\partial\lambda_j} + [3d+4+2a(d^2+3d+3)]\sum_i \lambda_i\frac{\partial P}{\partial\lambda_i} + (a+1)\sum_{i\neq j}\frac{\lambda_i\lambda_j}{\lambda_i-\lambda_j}\left(\frac{\partial P}{\partial\lambda_i} - \frac{\partial P}{\partial\lambda_j}\right) + \frac{1}{2}d(d+1)(d+2)\times[1+a(d+1)]P \tag{8}$$

(from here on the tildes are dropped). Among the solutions of this equation, those corresponding to the PDF must be non-negative, finite, and normalizable. The normalization is as follows [17]:

$$\int d\lambda_1 \cdots d\lambda_d P(\lambda_1,\ldots,\lambda_d) \prod_{i<j} |\lambda_i - \lambda_j| = 1. \qquad (9)$$

Clearly, the original stochastic equation (2) preserves the signature of the metric. We will restrict ourselves to the case of all positive λ's. Since there is no means of distinguishing between different orderings of the eigenvalues, the PDF must be a symmetric function with respect to all permutations of $\lambda_1,\ldots,\lambda_d$.

Now notice that in the logarithmic variables $z_i = \ln(\lambda_i)$ the master equation (8) describes the dynamics of d pairwise interacting particles on the line. Furthermore, one can consider those dynamics in the reference frame associated with the center of mass of the particles $z = d^{-1}\Sigma z_i$. Upon denoting the coordinates of the particles in this frame by $\zeta_i = z_i - z$ and noticing that $\det(\hat{g}) = g = \exp(zd)$, one finds that P now satisfies

$$\partial_t P = d[1+a(d+1)]\left(2g^2 \frac{\partial^2 P}{\partial g^2} + (2d+5) g \frac{\partial P}{\partial g} + \frac{1}{2}(d+1)\right.$$
$$\left. \times (d+2)P\right) + 2(1+a)\left[-\frac{1}{d}\sum_{i,j}^{d} \frac{\partial^2 P}{\partial \zeta_i \partial \zeta_j} + \sum_{i=1}^{d} \frac{\partial^2 P}{\partial \zeta_i^2}\right.$$
$$\left. + \frac{1}{2}\sum_{i<j}^{d} \frac{1}{\tanh\frac{1}{2}(\zeta_i - \zeta_j)}\left(\frac{\partial P}{\partial \zeta_i} - \frac{\partial P}{\partial \zeta_j}\right)\right], \qquad (10)$$

where the d variables $\zeta_1,\ldots,\zeta_d$ are not independent since $\Sigma \zeta_i = 0$. The Hamiltonian remaining after the dynamics of the center of mass are separated is translationally invariant: therefore, the total momentum of the particles, $\Sigma(\partial P/\partial \zeta_i)$, is conserved. The normalization rule now is

$$\int d\zeta_1 \cdots d\zeta_d \delta(\zeta_1 + \cdots + \zeta_d) |J(\zeta)| \int dg\, g^{(d-1)/2} P = 1,$$

$$J(\zeta) = \frac{2^{d(d-1)/2}}{d} \prod_{i<j}^{d} \sinh\frac{1}{2}(\zeta_i - \zeta_j), \qquad (11)$$

where we denote the set $\{\zeta_1,\ldots,\zeta_d\}$ by ζ. The operator in the square brackets in Eq. (10) is a Calogero-Sutherland Hamiltonian $\tilde{H}_S$, which is exactly solvable (see, e.g., [14,18,19]; this Hamiltonian appeared in a similar context in [7,13]). The Hamiltonian $\tilde{H}_S$ is the same for covariant and contravariant tensors, and in both Eulerian and Lagrangian frames.

It is important that $\tilde{H}_S$ is *self-adjoint* with respect to the measure (11). Its eigenfunctions are the so-called Jack polynomials, which are homogeneous polynomials in $\exp(\zeta_i)$ and are symmetric with respect to all permutations of ζ_i. Their construction is discussed in Appendix C. We will use these particular eigenfunctions of this operator in Secs. V and VI.

One sees that if P is initially chosen in the factorized form $P = P_1(g) P_2(\zeta_1,\ldots,\zeta_d)$, it will remain so factorized at all times. Thus the statistics of g are independent of the statistics of the ζ's at all times if they are initially independent. In particular, this property of Eq. (10) allows one to consider separately the PDF's for the determinant of the metric and for the logarithmic quantities $\zeta_i = \ln(\lambda_i g^{-1/d})$:

$$S(t,g) = \int d^d\zeta |J(\zeta)| P(g,\zeta),$$

$$F(t,\zeta) = \int dg\, g^{(d-1)/2} P(g,\zeta). \qquad (12)$$

An additional symmetry emerges in this context: Equations (8) and (11) remain invariant if the coordinates z_i of all particles are simultaneously *reflected* with respect to their center of mass. Such reflection leaves the center of mass intact and reverses the signs of all ζ_i, i.e., transforms all λ_i into $g^{2/d}/\lambda_i$. The origin of this symmetry can be understood if one notices that the master equations for the PDF's of the dimensionless quantities $G_{ik} = g^{-1/d} g_{ik}$ and $G^{ik} = g^{1/d} g^{ik}$ are the same, although the initial stochastic equations are different. This symmetry leads to nontrivial results for $d \geqslant 3$ and will be considered in Sec. IV.

III. EULERIAN AND LAGRANGIAN PDF's

The equation for the metric-determinant PDF $S(t,g)$ follows from Eq. (10):

$$\partial_t S = 2g^2 \frac{\partial^2 S}{\partial g^2} + (2d+5) g \frac{\partial S}{\partial g} + \frac{1}{2}(d+1)(d+2) S, \qquad (13)$$

where we have rescaled time by the factor $\gamma = d[1 + a(d+1)]$. This factor is always non-negative and vanishes if the velocity field is incompressible, $a = -1/(d+1)$, in which case any time-independent function $S(g)$ is a solution. Note that the right-hand side of Eq. (13) becomes a full derivative when multiplied by the Jacobian $g^{(d-1)/2}$. The solution of this equation is a log-normal distribution:

$$S(t,g) = \frac{g^{-(d+1)/2}}{\sqrt{8\pi\gamma t}} \exp\left(-\frac{[\ln(g) + \gamma t]^2}{8\gamma t}\right), \qquad (14)$$

where we took the initial distribution in the form $S(0,g) = \delta(g-1)$.

This result can be simply understood if one notes that the determinant g obeys the same equation as ρ^2, the squared density of the medium. The density satisfies the continuity equation, which can be written in logarithmic form:

$$\partial_t \ln\rho + \xi^k \partial_k \ln\rho + \xi^k_{,k} = 0. \qquad (15)$$

Since the time increments of ξ^k are independent identically distributed random variables, the central limit theorem implies the normal distribution of $\ln\rho$. Indeed, either from Eq. (13) or directly from Eq. (15), one can easily establish that the density PDF $R(t,\rho) = 2\rho^d S(t,\rho^2)$ satisfies $\partial_t R = (\gamma/2)(\rho^2 R)''$.

So far, we have worked in an Eulerian frame, considering statistics at an arbitrary fixed point $\mathbf{x}$. Now we show how the one-point Eulerian and Lagrangian PDF's are related. Let us assume that initially Lagrangian particles are uniformly distributed in space. We denote the Eulerian PDF by $P_E(\rho,\zeta;t,\mathbf{x})$; the Lagrangian PDF by $P_L(\rho,\zeta;t,\mathbf{y})$, where $\mathbf{y}$ is the Lagrangian label (initial coordinate of the Lagrangian particle); and the density of the medium by $\rho = |\det(\partial \mathbf{y}/\partial \mathbf{x})|$. The relation between P_E and P_L can be established from the following:

$$P_E(\rho,\zeta;t,\mathbf{x}) = \langle \delta(\rho-\rho(t,\mathbf{x}))\delta(\zeta-\zeta(t,\mathbf{x}))\rangle = \int \frac{d\mathbf{y}}{\rho}\langle \delta(\mathbf{x}-\mathbf{x}(t,\mathbf{y}))\delta(\rho-\rho(t,\mathbf{y})) \times \delta(\zeta-\zeta(t,\mathbf{y}))\rangle. \quad (16)$$

Since the one-point PDF $P_E(\rho,\zeta;t,\mathbf{x})$ is independent of position (due to spatial homogeneity), one can integrate Eq. (16) with respect to $\mathbf{x}$. Also noting that the one-point PDF $P_L(\rho,\zeta;t,\mathbf{y})$ is independent of $\mathbf{y}$, one gets

$$P_E(\rho,\zeta) = \frac{1}{\rho}P_L(\rho,\zeta). \quad (17)$$

Transformation to the Lagrangian frame can also be performed at the level of the original stochastic equations such as Eq. (15) with the aid of the stochastic calculus (see, e.g., [17] or [20]).

If one chooses initially $S(0,g) \propto \delta(g-\rho_0^2)$, one may substitute $\rho = \sqrt{g}$ in formula (17). One sees therefore that only the PDF of g is affected by the transformation between Eulerian and Lagrangian frames. The Lagrangian version of $S(g)$ is

$$S(t,g) = \frac{g^{-(d+1)/2}}{\sqrt{8\pi\gamma t}}\exp\left(-\frac{[\ln(g)-\gamma t]^2}{8\gamma t}\right). \quad (18)$$

Analogous results for the contravariant case are presented in Appendix A.

The log-normal statistics such as Eqs. (14) and (18) are a signature of this problem, and they will also be present for fluctuations of the eigenvalue ratios in asymptotically free regimes, i.e., where different ratios do not interact with each other [3–8].

IV. PDF's OF EIGENVALUE RATIOS IN TWO AND THREE DIMENSIONS

We saw in the previous section that $F(t,\zeta)$, the PDF of the ratios $\lambda_i/g^{1/d}$, would remain the same in both Eulerian and Lagrangian frames. In this section we analyze the equations for these PDF's in the two-and three-dimensional cases. Having in mind numerical simulations, we will write these equations using $d-1$ *independent* variables. In the general case such reduction is done in Appendix B.

Let us start with the two-dimensional case. It is now convenient to integrate the δ function in Eq. (11) and work with the logarithm of the eigenvalue ratio as a new variable: $x = \frac{1}{2}\ln(\lambda_1/\lambda_2) = \frac{1}{2}(\zeta_1-\zeta_2)$. The equation for $F(t,x)$ then becomes

$$\partial_t F = (1+a)\left[F''_{xx} + \frac{1}{\tanh(x)}F'_x\right]. \quad (19)$$

As expected, the right-hand side of Eq. (19) becomes a full derivative when multiplied by the Jacobian $J(x) = 2\sinh(x)$. Note that the differential operator on the right-hand side of Eq. (19) becomes a Legendre operator under the change of variables $\tilde{x} = \cosh(x)$. This property is a consequence of integrability of the initial Hamiltonian $\tilde{H}_S$ [Eq. (10)] and will be of use in Sec. V when one calculates the moments of passive vectors.

The nature of the solution can be easily understood if one first considers only the advective term $F'_x/\tanh(x)$. The characteristic of Eq. (19) then satisfies $\dot{x} = 1/\tanh(x)$, which implies that F is advected to regions where $|x| \geqslant 1$ and, for $t \to \infty$, that the asymptotic solution can be found from Eq. (19) by approximating $\tanh(x) \approx 1$. The asymptotic is log-normal as expected.

Note that the reflection symmetry $x \to -x$ of Eq. (19) is just a consequence of the previously mentioned general symmetry $\lambda_1 \leftrightarrow \lambda_2$ and does not add anything new. The function F must be *initially* chosen in such a symmetric form. This is not so in the three-dimensional case, which we now consider in more detail.

In three dimensions, upon integrating the delta function in Eq. (11) as before and introducing the new variables $x = \frac{1}{2}\ln(\lambda_1/\lambda_3)$ and $y = \frac{1}{2}\ln(\lambda_2/\lambda_3)$, one obtains the equation for $F(t,x,y)$:

$$\partial_t F = (1+a)\Bigg[F''_{xx} + F''_{xy} + F''_{yy} + \left(\frac{1}{\tanh(x)} + \frac{\sinh(x)}{2\sinh(y)\sinh(x-y)}\right)F'_x + \left(\frac{1}{\tanh(y)} + \frac{\sinh(y)}{2\sinh(x)\sinh(y-x)}\right)F'_y\Bigg]. \quad (20)$$

The normalization Jacobian for this PDF is $J(x,y) = \frac{32}{3}\sinh(x)\sinh(y)\sinh(x-y)$.

The symmetry with respect to all permutations of eigenvalues $\lambda_1,\lambda_2,\lambda_3$ leads to the following two symmetries of the solutions of Eq. (20):

$$x \to -x, \quad y \to y-x, \quad \text{and } x \leftrightarrow y. \quad (21)$$

Equation (20) posesses another (reflection) symmetry as well:

$$x \to -x, \quad y \to -y, \quad (22)$$

which corresponds to the inversion of $\lambda_1/\lambda_3, \lambda_2/\lambda_3$, and does not follow from Eq. (21). Therefore a general initial distribution should contain both symmetric, F^s, and antisymmetric, F^a, parts with respect to this reflection. The symmetries (21) act as reflections (22) on the points of the plane located on the lines $y=2x$, $y=x/2$, and $y=-x$; hence, the antisymmetric part of the PDF F^a must vanish on these lines.

Characteristic trajectories of Eq. (20) are presented in Fig. 1. The lines $y=\pm x$, $y=2x$, $y=x/2$, $x=0$, and $y=0$ are combined in groups that are transformed by the symmetries (21) independently. Those groups correspond to sheet, tube, and strip volume deformations as shown. Let us concentrate our attention on the sector $x \geqslant 0$, $y \leqslant 0$. Due to the symmetries (21) and (22), this allows one to understand the behavior of the PDF in the entire $(x,\, y)$ plane. Considering the characteristic trajectories (they advect F towards the line $y = -x$ from both sides) or the flux of the conserved function

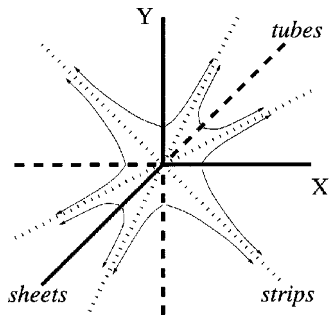

FIG. 1. Characteristic trajectories of Eq. (20). Solid lines correspond to sheet configurations, dashed lines to tubes, and dotted lines to strips.

$F(x,y)|J(x,y)|$ (calculated on the line $y=-x$, it is found to be directed from the semisector with positive F^a to that with negative F^a), one can show that the antisymmetric part of the PDF decays with time. The symmetry of the solution with respect to the sheet and tube configurations thus emerges asymptotically as $t\to\infty$. Again, we emphasize that this conclusion holds only until diffusion comes into play.

Numerical solutions of Eq. (20) with various initial distributions concentrated in the region $|x|\leqslant 1$, $|y|\leqslant 1$, confirm that the PDF becomes symmetrized very fast, at times $t\sim 1$. In the region $|x|\gg 1$, $|y|\gg 1$, far from the lines $y=x$, $y=0$, and $x=0$, the long-time ($t\gg 1$) asymptotic is log-normal. This asymptotic can be easily obtained from Eq. (20).

V. PASSIVE VECTORS

In this section we apply the developed formalism to passively advected vectors. Consider the evolution of the coordinates of a particle advected by the fluid: $x^i=x^i(t,\mathbf{y})$, where y^i is the initial position of the particle, i.e., $x^i(0,\mathbf{y})=y^i$. An infinitesimal contravariant vector a^i changes under such coordinate transformation as follows: $a^i(t,\mathbf{x})=(\partial x^i/\partial y^k)a_0^k(\mathbf{y})$. In order to find the mean of any object constructed out of a^i, we have to average it with respect to the initial distribution of $a_0^i(\mathbf{y})$ and with respect to all realizations of the random velocity field ξ^k. The latter averaging can be done via the PDF's for covariant and contravariant (metric) tensors. Let us assume that the initial distribution of the vector a^i is Gaussian, isotropic, and independent of $\mathbf{y}$: $\langle a_0^i a_0^j\rangle=\delta^{ij}$. As an example, consider the moments $A_n=\langle|\mathbf{a}|^{2n}\rangle$:

$$A_n=\langle(\mathbf{a}_0\cdot\hat{g}\cdot\mathbf{a}_0)^n\rangle, \tag{23}$$

where $\hat{g}$ is the *contravariant* tensor advected by the fluid and with the initial condition $g^{ij}(0,\mathbf{y})=\delta^{ij}$. The distribution of this tensor can be found in the same way as that of the covariant tensor and is discussed in Appendix A. Moments of a *covariant* vector a_i can be found using exactly the same formula (23), with $\hat{g}$ now being the *covariant* tensor.

It is easy to see that the average (23) is a sum of terms of the form $\langle \mathrm{Tr}^k(\hat{g}^l)\rangle$, where $kl=n$. To simplify the formula (23), we note that the eigenvalues of the matrix $\hat{g}$ can be expressed as $\lambda_i=g^{1/d}\exp(\zeta_i)$. Therefore, for all n, $g^{-n/d}\mathrm{Tr}(\hat{g}^n)$ depends only on ζ_i and is independent of the determinant g. Since the initial distribution of g is $S(g)=\delta(g-1)$, one can average powers of g independently to obtain

$$A_n=f_d(n,t)\langle g^{n/d}\rangle,$$
$$f_d(n,t)=\langle g^{-n/d}(\mathbf{a}_0\cdot\hat{g}\cdot\mathbf{a}_0)^n\rangle, \tag{24}$$

where the functions $f_d(n,t)$ do not depend on the statistics of the determinant and are, therefore, *universal*. These functions are the same in the covariant and contravariant cases, and in both Eulerian and Lagrangian frames. The only parts of the moments A_n that are nonuniversal are the averages of the determinant. These averages can be calculated exactly using formulas (14), (18), and (A3):

$$\langle g^s\rangle_E^{\mathrm{co}}=\langle g^s\rangle_L^{\mathrm{contra}}=e^{s(2s-1)\gamma t}, \tag{25}$$

$$\langle g^s\rangle_L^{\mathrm{co}}=\langle g^s\rangle_E^{\mathrm{contra}}=e^{s(2s+1)\gamma t}. \tag{26}$$

The universal functions $f_d(n,t)$ can, in fact, be easily calculated directly (cf. [6,7,21,22]) if one starts from the equation for advection of a passive vector $a^i(t,\mathbf{x})$:

$$\partial_t a^i+\xi^k a^i_{,k}-\xi^i_{,k}a^k=0, \tag{27}$$

where statistics of $\xi^i(t,\mathbf{x})$ are given by Eq. (3). However, for methodical purposes, we prefer to rederive this result using the technique of Jack polynomials. In addition to also being quite simple, it furthermore illustrates the general method that can be applied to finding moments of *any* passively advected tensor. In Sec. VI, we show, e.g., how moments of a bilinear form $a^i b^k$ can be calculated.

Formula (24) can be further simplified if one does the average with respect to the distribution of a_0^i. Upon introducing the generating function

$$Z(\beta)=\langle\exp[\beta g^{-1/d}(\mathbf{a}_0\cdot\hat{g}\cdot\mathbf{a}_0)]\rangle, \tag{28}$$

one can represent f_d as

$$f_d(n,t)=\left[\frac{\partial^n Z(\beta)}{\partial\beta^n}\right]_{\beta=0}. \tag{29}$$

The Gaussian average with respect to the initial distribution of the vector can now be done easily, resulting in

$$Z(\beta)=\left\langle\prod_{i=1}^{d}[1-\beta\exp(\zeta_i)]^{-1/2}\right\rangle, \tag{30}$$

where the remaining averaging is with respect to the statistics of ζ_i. The PDF of the ζ's is $F(\zeta)|J(\zeta)|\delta(\Sigma\zeta_i)$ with the initial condition $\delta(\zeta_1)\cdots\delta(\zeta_d)$. It is important that the function that is being averaged in Eq. (30) is the generating function for the particular class of Jack polynomials that are eigenfunctions of the self-adjoint Calogero-Sutherland operator $\tilde{H}_S$ in Eq. (10). Therefore, all functions (29) can be found exactly in the general case. The appropriate calculation is carried out in Appendix C. The answer is

$$f_d(n,t)=\left(\frac{d}{2}\right)_n\exp\left[\left(\frac{d-1}{d}\right)n(2n+d)(1+a)t\right], \tag{31}$$

where we use the Pochhammer notation $(d/2)_n=(d/2)(d/2+1)\cdots(d/2+n-1)$.

In the two-dimensional case, the corresponding result can be obtained in a rather simple manner, which nevertheless illustrates the main idea of the general derivation. In order to do this, notice that the generating function $Z(\beta)$, expressed in the two-dimensional case in terms of $x=\frac{1}{2}(\zeta_1-\zeta_2)$ (see Sec. IV), coincides with the generating function for the Legendre polynomials $P_n(\cosh(x))$ and, therefore, $f_2(n,t)=n!\langle P_n(\cosh(x))\rangle$. The average can now be completed with the aid of Eq. (19). Upon multiplying it by $|J(x)|P_n(\cosh(x))$, integrating by parts twice, and using the equation for the Legendre polynomials, $P_n''(\mu)(\mu^2-1)+2\mu P_n'(\mu)=n(n+1)P_n(\mu)$, one gets

$$\dot{f}_2(n,t)=(1+a)n(n+1)f_2(n,t),$$
$$f_2(n,t)=n!\exp[n(n+1)(1+a)t], \quad (32)$$

which is in agreement with Eq. (31).

As an example, consider moments of a magnetic field advected by the fluid. The contravariant vector in this case is B^i/ρ, where ρ is the density of the fluid. Let us denote the moments of B^i as $H_n=\langle|\mathbf{B}|^{2n}\rangle$. Upon recalling that in the contravariant case $g=1/\rho^2$, one gets from Eq. (24)

$$H_n=f_d(n,t)\langle g^{-n(d-1)/d}\rangle^{\text{contra}}, \quad (33)$$

where for the g average one uses the formula (26) in the Eulerian frame or formula (25) in the Lagrangian frame.

An analogous derivation can be carried out for a covariant vector, e.g., the gradient of a passive scalar $\nabla\theta$. For its moments $C_n=\langle|\nabla\theta|^{2n}\rangle$, one finds

$$C_n=f_d(n,t)\langle g^{n/d}\rangle^{\text{co}}, \quad (34)$$

where for the g average one uses formula (25) or (26) depending on the frame of reference.

VI. PASSIVE TENSORS OF ANY RANK

We now briefly demonstrate how one can calculate exactly the moments of a passively advected higher-rank tensor $\hat{T}$. Suppose that one is interested in some moment $\langle\hat{T}^m\rangle$. After averaging with respect to the initial distribution of $\hat{T}$, one is left with a combination of $\text{Tr}^k(\hat{g}^n)$, which are polynomials of degree nk in the eigenvalues of the metric $\hat{g}$. But any symmetric polynomial of degree m can be expanded in Jack polynomials of degree m, which can then be averaged exactly. The result will therefore be a linear combination of exponents growing at the rates given by Eq. (C10).

For example, consider a contravariant bilinear form a^ib^k, where a^i and b^k are initially independent Gaussian random vectors, $\langle a_0^ia_0^k\rangle=\langle b_0^ib_0^k\rangle=\delta^{ik}$, and find its second moment $B_2=\langle(\mathbf{a}\cdot\mathbf{b})^2\rangle=\langle\text{Tr}(\hat{g}^2)\rangle$. Then,

$$B_2=\left\langle\sum_i^d\lambda_i^2\right\rangle=\langle g^{2/d}\rangle\left(\langle J_{(2,0)}\rangle-\frac{2}{3}\langle J_{(1,1)}\rangle\right), \quad (35)$$

where the polynomials $J_{(2,0)}$ and $J_{(1,1)}$ are constructed in Eq. (C4). The corresponding eigenvalues are $\tilde{E}^{(2)}_{(2,0)}=(d+4)(d-1)/d$ and $\tilde{E}^{(2)}_{(1,1)}=(d^2-4)/d$, as follows from Eq. (C10). The answer is

$$B_2=\exp\left[\left(\frac{8}{d^2}+\frac{2}{d}\right)\gamma t\right]\left[\left(\frac{d^2+2d}{3}\right)\exp[2\tilde{E}^{(2)}_{(2,0)}(1+a)t]-\left(\frac{d^2-d}{3}\right)\exp[2\tilde{E}^{(2)}_{(1,1)}(1+a)t]\right]. \quad (36)$$

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APPENDIX A: PDF FOR THE CONTRAVARIANT TENSOR

The derivation of the main equations for the case of a contravariant tensor is quite similar to the case of the covariant tensor. Here we just explain the origin of the difference and write out the main results. The dynamical equation for the contravariant case reads

$$\partial_t\varphi^{ij}+\xi^k\varphi^{ij}_{,k}-\xi^i_{,k}\varphi^{kj}-\xi^j_{,k}\varphi^{ik}=0. \quad (A1)$$

The derivation of the master equation can be carried out the same way as in the covariant case and results in different coefficients in the g part of Eq. (10). The ζ part remains intact. This is not surprising, since the transition from φ_{ij} to φ^{ij} does not change the ratios of the eigenvalues, but results only in the inversion of the determinant: $g\to1/g$. Accordingly, the equation for the PDF of a contravariant tensor, $\tilde{P}(\tilde{g},\zeta)$, can be obtained from Eq. (10) by substituting $P=\tilde{P}\tilde{g}^{d+1}$, $g=1/\tilde{g}$. The resulting equation for the PDF of the determinant, $\tilde{S}(\tilde{g})$, is (dropping the tildes):

$$\partial_tS=2g^2\frac{\partial^2S}{\partial g^2}+(2d+3)g\frac{\partial S}{\partial g}+\frac{1}{2}d(d+1)S, \quad (A2)$$

where time has been rescaled by the factor of γ as in Sec. III. When using this equation, one should remember that g now satisfies the same equation as $1/\rho^2$, where ρ is the density of the medium. Equation (A2) is written in the Eulerian frame. For completeness, we write down the solution of Eq. (A2) with initial distribution $S(0,g)=\delta(g-1)$:

$$S(t,g)=\frac{g^{-(d+1)/2}}{\sqrt{8\pi\gamma t}}\exp\left(-\frac{[\ln(g)-\gamma t]^2}{8\gamma t}\right). \quad (A3)$$

The Lagrangian analog of Eq. (A3) is obtained via multiplication by $\rho=1/\sqrt{g}$.

APPENDIX B: PDF's OF EIGENVALUE RATIOS

The δ function Eq. in (11) can be integrated over, and $\zeta_1,\ldots,\zeta_d$ can be reduced to $d-1$ *independent* variables, viz., the logarithms of the eigenvalue ratios: $x_n=\frac{1}{2}\ln(\lambda_n/\lambda_d)=\frac{1}{2}(\zeta_n-\zeta_d)$. In these variables, the equation for F becomes

$$\partial_t F=(1+a)\Bigg\{\sum_{n=1}^{d-1}\frac{\partial^2 F}{\partial x_n^2}+\sum_{n=1}^{d-1}\frac{1}{\tanh(x_n)}\frac{\partial F}{\partial x_n}+\frac{1}{2}\sum_{n\neq m}^{d-1}\frac{\partial^2 F}{\partial x_n\partial x_m}+\frac{1}{4}\sum_{n\neq m}^{d-1}\frac{1}{\sinh(x_n-x_m)}\times\left[\frac{\sinh(x_n)}{\sinh(x_m)}\frac{\partial F}{\partial x_n}-\frac{\sinh(x_m)}{\sinh(x_n)}\frac{\partial F}{\partial x_m}\right]\Bigg\}. \tag{B1}$$

The last two terms correspond to interactions between different x's and only enter for $d\geqslant 3$. The normalization rule now is

$$\frac{2^{(d+2)(d-1)/2}}{d}\int\prod_{n<m}^{d-1}|\sinh(x_n-x_m)|\prod_{n=1}^{d-1}|\sinh(x_n)|dx_nF=1. \tag{B2}$$

This form of the equation for F is most convenient for numerical solution and for geometric analysis such as that of Sec. IV.

APPENDIX C: JACK POLYNOMIALS

Jack polynomials [23] $J_\mu(x_1,\ldots,x_d;\alpha)$ of degree m are homogeneous (of degree m) polynomials, depending on d variables x_i, and symmetric under all permutations of x_i. They depend on a parameter α and are labeled by partitions μ of an integer number m.

The partition μ of m is a nonincreasing sequence of integers: $\mu=(\mu_1\geqslant\cdots\geqslant\mu_d)\in\mathbb{Z}^d_{\geqslant 0}$, such that $m=\mu_1+\cdots+\mu_d$. The polynomials $J_\mu(x;\alpha)$ vanish if the number of parts $l(\mu)$ is greater than the number of variables d. Consider two partitions μ and λ of the same length $l(\mu)=l(\lambda)=d$. One writes that $\lambda\geqslant\mu$ if $\lambda_1+\cdots+\lambda_i\geqslant\mu_1+\cdots+\mu_i$ for each $i\leqslant d$. This defines the so-called *natural* (or *dominance*) ordering on partitions.

In order to give a formal definition of the Jack polynomials, first define the *monomial symmetric function* m_μ corresponding to the partition μ:

$$m_\mu=\sum x_1^{\mu_1}\cdots x_d^{\mu_d}, \tag{C1}$$

where the summation is over all permutations of $\mu_1,\ldots,\mu_d$. The Jack polynomials must, by definition, be represented as

$$J_\lambda(x;\alpha)=\sum_{\mu\leqslant\lambda}u_{\lambda\mu}m_\mu \tag{C2}$$

and be the eigenfunctions of the Calogero-Sutherland Hamiltonian

$$H_S^{(\alpha)}=\sum_{i=1}^{d}\left(x_i\frac{\partial}{\partial x_i}\right)^2+\frac{2}{\alpha}\sum_{i\neq j}^{d}\frac{x_i^2}{x_i-x_j}\frac{\partial}{\partial x_i}. \tag{C3}$$

All coefficients $u_{\lambda\mu}$ can be found recursively in terms of $u_{\lambda\lambda}$ with the aid of this definition [23].

Let us use this definition to construct the Jack polynomials for $m=2$ and $\alpha=2$. The corresponding partitions are (2,0,0,...) and (1,1,0,...). Using the first condition (C2), one writes

$$J_{(2,0)}(x;2)=\sum_{i=1}^{d}x_i^2+A\sum_{i<j}^{d}x_ix_j,$$

$$J_{(1,1)}(x;2)=\sum_{i<j}^{d}x_ix_j. \tag{C4}$$

The coefficient A must be found from the requirement that the polynomials be eigenfunctions of Eq. (C3), which gives $A=2/3$.

The eigenvalues (energies) corresponding to Jack polynomials are

$$E_\mu^{(\alpha)}=\sum_{i=1}^{d}\mu_i^2+\frac{2}{\alpha}\sum_{i=1}^{d}(d-i)\mu_i. \tag{C5}$$

The energies (C5) depend on particular partitions μ. Any symmetric polynomial of degree m can be expanded in Jack polynomials of the same degree m. Of all the other properties of the Jack polynomials, we will need

$$\prod_{i,j}^{d}\frac{1}{(1-x_iy_j)^{1/\alpha}}=\sum_\mu b_\mu(\alpha)J_\mu(x;\alpha)J_\mu(y;\alpha), \tag{C6}$$

where the summation is performed over all possible partitions μ of all non-negative integers, and $b_\mu(\alpha)$ are expansion coefficients that can be found in [23]. For present purposes, we will need the formula (C6) with the set $\{y_j\}$ consisting of only one variable. In this case the expansion takes the form

$$\prod_{i=1}^{d}\frac{1}{(1-yx_i)^{1/\alpha}}=\sum_{m=0}^{\infty}y^mQ_{(m)}(x;\alpha), \tag{C7}$$

where $\mu=(m)$ is a partition consisting of only one element. $Q_{(m)}(x;\alpha)$ stands for the properly normalized Jack polynomials. The explicit expression for $Q_{(m)}(x;\alpha)$ is

$$Q_{(m)}(x;\alpha)=\sum_{1\leqslant i_1\cdots\leqslant i_m}^{d}\frac{(\theta)_{q_1}\cdots(\theta)_{q_d}}{q_1!\cdots q_d!}x_{i_1}x_{i_2}\cdots x_{i_m}, \tag{C8}$$

where $\theta=1/\alpha$, $q_l=\#\{n|i_n=l\}$ is the multiplicity with which the number $l=1,2,\ldots,d$ appears in $i_1\cdots i_m$, and $(\theta)_q=\theta(\theta+1)\cdots(\theta+q-1)$.

To use these results one needs to transform the Hamiltonian [in the square brackets in Eq. (10)] to the form (C3). Changing variables to $\tilde{\lambda}_i=\exp(\zeta_i)=\lambda_i g^{-1/d}$, one gets

$$\begin{aligned}\tilde{H}_S=&-\frac{1}{d}\sum_{i,j}^{d}\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}+\sum_{i=1}^{d}\frac{\partial^2}{\partial\zeta_i^2}\\&+\frac{1}{2}\sum_{i<j}^{d}\frac{1}{\tanh\frac{1}{2}(\zeta_i-\zeta_j)}\left(\frac{\partial}{\partial\zeta_i}-\frac{\partial}{\partial\zeta_j}\right)\\=&H_S^{(2)}-\frac{1}{d}\left(\sum_{i=1}^{d}\tilde{\lambda}_i\frac{\partial}{\partial\tilde{\lambda}_i}\right)^2-\left(\frac{d-1}{2}\right)\sum_{i=1}^{d}\left(\tilde{\lambda}_i\frac{\partial}{\partial\tilde{\lambda}_i}\right).\end{aligned} \tag{C9}$$

For any Jack polynomials of degree m, the corresponding eigenvalues $\tilde{E}_\mu$ of the Hamiltonian (C9) are

$$\tilde{E}_\mu=E_\mu^{(2)}-\frac{m^2}{d}-\frac{(d-1)}{2}m. \tag{C10}$$

In particular, the energy of $Q_{(m)}(\tilde{\lambda};2)$ is

$$\tilde{E}_m=\left(\frac{d-1}{d}\right)m\left(m+\frac{d}{2}\right). \tag{C11}$$

Now notice that the averaged Jack polynomials $Q_{(n)}(\tilde{\lambda};2)$ and the functions $f_d(n,t)/n!$ have the same generating function [see Eqs. (30) and (C7)], whence

$$f_d(n,t)=n!\langle Q_{(n)}(\tilde{\lambda};2)\rangle. \tag{C12}$$

Since [Eq. (10)] $\partial_t F=2(1+a)\tilde{H}_S F$, where $\tilde{H}_S$ is self-adjoint with respect to the measure (11), $f_d(n,t)$ satisfies $\dot{f}_d(n,t)=2(1+a)\tilde{E}_n f_d(n,t)$, the solution of which (with correct initial condition) is the expression (31).

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